Models of Set Theory I – Summer 2017

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Problem 33 [6 points]

- An *interval partition of* ω is a partition of ω into finite nonempty intervals I_n, for $n \in \omega$, ordered naturally. That is, ω is the disjoint union of the I_n, $\min I_0 = 0$, and $\min I_{i+1} = \max I_i + 1$ for every $i < \omega$.
- A *chopped real* is a pair (x, Π), where $x \in {}^\omega 2$ is a real, and Π is an interval partition of ω. A real $y \in {}^\omega 2$ *matches* a chopped real (x, Π) if $x \restriction I = y \restriction I$ (we also say that x and y *agree* on I) for infinitely many intervals $I \in \Pi$.
- A set X of reals is *nowhere dense* if for every $r \in {}^{<\omega}2$ there is $s \in {}^{<\omega}2$ such that $s \supseteq r$ and $I_s \cap X = \emptyset$. A set of reals is *meager* if it is the countable union of nowhere dense sets.

Add the missing steps, and fill in additional details, in the below proof sketch of the following **Theorem:** If $M \subseteq {}^\omega 2$ is meager, then there is a chopped real that is not matched by any element of M.

Proof sketch: Suppose M is meager, fix nowhere dense sets $\langle F_n \mid n < \omega \rangle$ such that $M = \bigcup F_n$. We may assume that the F_n are $\subseteq$-increasing (why?). We construct a chopped real $(x, \langle I_n \mid n \in \omega \rangle)$ such that for each n, no real in F_n agrees with x on I_n. This suffices (why?).

To define I_n and $x \restriction I_n$, suppose the earlier I_k and $x \restriction \bigcup_{k<n} I_k$ are already defined. Let $m = \bigcup_{k<n} I_k$. I_n will be the union of 2^m (naturally ordered) subintervals J_i, for $i < 2^m$, defined as follows. Let $\langle u_i \mid i < 2^m \rangle$ enumerate all functions from m to 2. By induction on i, choose J_i and $x \restriction J_i$ so that no element of F_n extends $u_i \cup \bigcup_{j \leq i} (x \restriction J_j)$ – why is this possible? Finally, let $I_n = \bigcup_{i<2^m} J_i$.

It follows that, for any n, if y agrees with x on I_n, then $y \notin F_n$ (why?), as desired.

Problem 34 [4 points] Let M be a countable ground model, and let P denote Cohen forcing. Show that after forcing with P over M, the ground model reals are not meager in any generic extension – add the missing steps, and fill in additional details, in the below proof sketch.

Proof sketch: Making use of Problem 33, we show that every chopped real (x, Π) in the Cohen extension is matched by some ground model real y. Given (x, Π), we construct a real y in M such that for any $p \in P$ and any natural number n, p does not force y to not agree with x on any interval I of Π that *starts beyond* n (that is $\min I \geq n$). This means that y is as desired (why?).

We build such y by a recursion of length ω, where each step defines $y(k)$ for finitely many k, and *takes care* of one pair (p, n). The latter means to extend p to a condition q forcing $x \restriction I$ to equal some $z \in M$ (see the second item of Problem 23 from Problem Sheet 6), for the first $I \in \Pi$ above n and above where we already have specified values for y. Now let $y \restriction I = z$.

Problem 35 [6 points] Let $\mathrm{Fn}(A, B, \kappa)$ denote the set of all functions f with $\mathrm{dom} f \subseteq A$, of size less than κ, and $\mathrm{ran} f \subseteq B$, ordered by setting $f \leq g$ iff $f \supseteq g$.

- Show that if $\dot{x}$ is a name for a real in $P = \mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)$ – the forcing to add λ-many Cohen reals, then there is a P-name $\dot{y}$ and a countable $X \subseteq \lambda$ such that $1_P \Vdash \dot{x} = \dot{y}$ and $\dot{y}$ is a $\mathrm{Fn}(X \times \omega, 2, \aleph_0)$-name.

Hint: Use that P satisfies the ω_1-cc, this was or will shortly be shown in the lecture, so you do not have to verify this property here in either case.

- Show that if X is countable, then $\mathrm{Fn}(X, 2, \aleph_0)$ is isomorphic to $\mathrm{Fn}(\omega, 2, \aleph_0)$.
- Show that if M is a countable ground model and G is $\mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)^M$-generic over M, then $G \cap Q$ is Q-generic over M whenever $Q = \mathrm{Fn}(X \times \omega, 2, \aleph_0)^M$ for some $X \subseteq \lambda$.
- Show that if M is a countable ground model, and $P = \mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)^M$, then any real in a P-generic extension of M is an element of a smaller generic extension of M, for the forcing to add a single Cohen real (that is $\mathrm{Fn}(\omega, 2, \aleph_0)$).
- Show that after adding λ-many Cohen reals over a countable ground model M, the ground model reals do not become meager.

Problem 36 [4 points] Let M be a countable ground model, and let B be a complete Boolean algebra in M. Let M^B be the Boolean-valued model as defined on Problem Sheet 8 (inside M). We define the forcing relation $p \Vdash \varphi(\dot{x}_1, \ldots, \dot{x}_n)$ if and only if $p \leq ||\varphi(\dot{x}_1, \ldots, \dot{x}_n)||$, for $p \in B \setminus \{0\}$.

- Verify that this relation agrees with the forcing relation for the forcing notion $B \setminus \{0\}$, as defined in the lecture course.
- Show that if G is an M-generic ultrafilter on B, then for all B-names $\dot{x}_1, \ldots, \dot{x}_n$, and all first order formulas φ,

$$M[G] \models \varphi(\dot{x}_1^G, \ldots, \dot{x}_n^G) \text{ if and only if } ||\varphi(\dot{x}_1, \ldots, \dot{x}_n)|| \in G.$$